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Matahematics education and the applicability of mathematics

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Real life applications enter primary and secondary school education in two ways – for creating interest in subjects which may otherwise be abstract, and for the purpose of making use of the school subjects in day-to-day situations. Here, the prime example is mathematics. A demand for a close connection between mathematics and applications in school may be found in national curricula, and is present in textbooks. On the other hand mathematics is considered and taught to be a deductive, a priori, science with internal truth makers, structured by propositions and proofs. Mathematics is presented both as empirically grounded and as an analytic science, creating a possible conflict for students. The problem of the applicability of mathematics is also discussed within philosophy of mathematics: How is it possible for a priori truths to contribute essentially to our descriptions of the world?

From a philosophical point of view, we try to shed light on how this seeming paradox may be explained and handled. Central are our views on mathematical concepts as explications and on concept formation in mathematics.

Keywords: abstraction, applicability, concept formation, explication, idealization, philosophy of mathematics education.

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Introduction

It is well known that personal interest and motivation on a student's behalf is a positive factor for learning.[1] In primary and secondary school this is often interpreted as a need for school subjects to be made relevant for the student in everyday life. For some subjects this may come natural enough, while for others the connection to the students' daily lives may be somewhat laboured. Several subjects include, already in primary school, abstract concepts that need concretising for the students to grasp. But the very concretisation may also be an obstacle for the student's understanding of the abstract. Here we discuss how to understand applicability, abstraction and generalisation from a philosophical point of view, taking mathematics as our prime example.

It may be argued that the road to mathematical knowledge goes via applied mathematics. Such a view is made explicit by Wright (2000) who, discussing a neo-Fregean foundation of analysis, declares that

> [...] it seems clear that one kind of access to e.g. simple truths of arithmetic precisely proceeds through their applications. Someone can – and our children surely typically do – first learn the concepts of elementary arithmetic by a grounding in their simple empirical applications and then, on the basis of the understanding thereby acquired, advance to an a priori recognition of simple arithmetical truths. (Wright, 2000, p. 265)

The question thus arises of how this empirical grounding is possible. How is it at all possible to ground a priori–truths empirically? In school, teachers are faced with the dilemma of, on the one hand, teaching mathematics as a general, abstract, a priori, deductive science, and, on the other hand, grounding and motivating students' interest in mathematics empirically through various types of applications. On the one hand mathematics is presented as based on proofs and having, so to speak, internal truth makers. On the other hand student motivation should, at least according to some curricula and educational approaches, be found in applications.

A question of concern is then how teachers may lift their mathematics teaching from the concrete everyday world to the general and abstract. As discussed by Witzke, Struve, Clark, and Stoffels (2016), referring to, among others, Schoenfeld (1985, 2011), the transition from the applied, near-empirical mathematics taught in lower grades to abstract, often axiomatic formal mathematics met at university may cause problems for students. This transition of view of the nature of mathematics, from the concrete to the abstract, may, according to them, even be a major reason for students to drop out before graduating in mathematics.

Lundin (2012), referring to among others, Dowling (2010), Palm (2002), and contributions in Verschaffel's et al (2009), discusses the related, though not altogether equivalent, issue of the use of word problems in mathematics education. As these authors show, the use of such problems, which is obviously connected to the application of mathematics, may be beneficial for learning, but may also be criticised. Lundin, however, takes a completely different stance from us in criticising the very idea of applicability of mathematics in a school context. Here we, on the contrary, show how the applicability of mathematics may be understood in such a context.

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[1] In the case of mathematics, cf. Hannula (2012) for a recent overview.

Now, the applicability of mathematics presents a well-known problem also in the philosophy of mathematics. A classical formulation is Wigner's:

> [...] the enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious and [...] there is no rational explanation for it. [...] The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. (Wigner, 1960, pp. 2, 14)

This problem is still in focus, which may be seen e.g. from Pincock's (2015) up to date overview in the Internet Encyclopedia of Philosophy. There Pincock also refers to Wright (2000), and, among others, Steiner (1998, 2005), Colyvan (2001), and Parsons (2008).[2] A recent example of the vitality of the current debate is also the on-going discussion of the Indispensability Thesis, emanating from Quine and Putnam, and later discussed by e.g. Maddy.[3] The thesis states that mathematics is not only useful or appropriate, but indispensable to science. Consequently, it is claimed, we should commit ourselves ontologically to mathematical entities, just as to physical ones. Learning mathematics is thus considered an indispensable part of learning science, and, as science, mathematics may in the light of its indispensability to science be considered empirically grounded.

Any standard empiricist view on knowledge and learning underlines the necessity of empirical grounding. Here one may, even taking Piaget's critique of empiricism into consideration, count both Piaget and Vygotskij as empiricists in the weak sense that knowledge, also in constructivist perspectives, is constructed in interaction with reality and is thus (in this weak sense) empirically grounded. A recent example arguing for empiricism in mathematics is Jenkins (2008). Mathematical concepts are according to her in general empirically grounded, in spite of mathematics being a priori.[4]

Ontologically, both nominalists and realists face the problem of the applicability of mathematics. Nominalists, who mean that there are no mathematical objects, have to explain how non-denoting terms such as "three" and "derivative" may be used with success. Realists, being of the opinion that mathematical objects are real but abstract, have to explain our knowledge of such objects, and their relation to the empirical world.[5] These two positions are not the only ones available, even if they are certainly the most common ones in the literature today. An alternative realist version is Aristotle's, and it may be argued that his views on the relation between mathematics and empirical reality provide a fruitful ground for an analysis

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[2] Steiner (1998, 2005) presents several quotations from researchers highlighting the same issue, and provides references from Plato through Descartes, Berkeley, and Kant to contemporary philosophers like Shapiro, Field, and Colyvan.

[3] There is a vast discussion on this issue amongst philosophers of mathematics. Cf. e.g. Quine (1981), Putnam (2012) and Maddy (1992, 1997). For up to date information and many more references cf. Colyvan's entry *Indispensability Argument* in www.plato.stanford.edu.

[4] Cf. Roland (2010), Tennant (2010), and Sjögren and Bennet (2014) for a discussion and criticism of Jenkins' views.

[5] This problem has been much discussed recently. Two different types of attack, are provided by Steiner (1998) and Bangu (2012).

of the problem of applicability.[6] The problem is thus not confined to mathematics education, but lives in a wider context.

It is common amongst philosophers of mathematics to view mathematical objects (entities) as abstract.[7] Here we adhere to this view, but do not commit ourselves to any particular ontology.

What we offer in this paper is a philosophical analysis of how mathematics relates to empirical reality. An understanding of this relation is, as we see it, a necessary component in understanding how, if at all, students from an early age may ground their mathematics in reality. Thus, it is our belief that the view we present may give guidance for mathematics education in providing the teacher with an awareness of the difficulties in question.

Note that the question we discuss is not an ontological one. On the contrary we wish to emphasize that our discussion should be taken as ontologically neutral. We see applicability as a relation between mathematics and empirical reality, whatever ontological status either of these realms are believed to have.

Next we provide a discussion on applications of mathematics in relation to learning. This is followed by our view of mathematical concepts as explications, in Carnap's sense, which leads us to a discussion of the abstract character of mathematical concepts, and the importance of making idealizations in applying mathematics. Finally, consequences for mathematics education are discussed.

In fact the choice of focusing on mathematics, rather than on some other abstract subject, is somewhat arbitrary. Thus we believe our discussion is relevant in a broader context than just mathematics and mathematics education.

Application Related to Learning

Without taking stance in any particular theory of learning, we outline some well-known examples of how mathematical concepts may be introduced to primary or secondary school students. Here we take negative numbers as a first example. Starting from the number line of natural numbers, negative numbers are introduced by teachers in various ways. Since the concept of negative number is abstract, and known to be difficult to grasp for some students, one normally starts off from some kind of concretization or metaphor.[8] Examples of concretizations and metaphors used in this case are negative numbers as vectors, as motion (balloons moving up or down and the like), as owing money, or as sad and happy people moving in or out of town.[9]

The idea behind using concretisations such as these is that the students should grasp the purpose of the metaphor used, and decontextualize, generalize and abstract to form their own conceptions of the mathematical concept in question, in this case *negative number*. Different theories of learning are used to explain this process, often underpinned

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[6] See Franklin (2014) for an elaboration of Aristotle's ideas, relevant for mathematics education.

[7] See the introduction by Panza and Sereni (2013) for some different opinions from both realists and nominalists on this issue.

[8] For an overview, and further references see Kilhamn (2011), who leans on Lakoff and Núñez (2000).

[9] See Kilhamn (2011, Ch. 3) for a vast number of different metaphors used. Here we do not discuss the difference between model, metaphor, and concretization. This is, however, also in this context an interesting subject, which will be discussed elsewhere.

by some form of constructivism.[10] The student is supposed to gain both interest and insight in the abstract concept via non-abstract examples from everyday life. There are, however, at least two problems connected with this way of reasoning.

First, it is not at all clear that fascination for a subject starts from applications in daily life. Examples of the contrary are subjects like astronomy or natural history. To our experience, many students from primary school and onwards have an interest e.g. in space or in dinosaurs. They may spend a lot of time collecting facts such as surface temperatures of stars and planets, distances in space or the life and habitats of animals extinct for sixty million years. Neither of these areas of knowledge connects easily to daily life. So why should mathematics need such a connection?

Second, there is a problem of abstraction. Apart from most subjects in school, mathematics is about abstract entities such as (negative) numbers or functions. These are not possible to point out ostensively like siblings, stars and (fossils of) dinosaurs. In fact this is the very reason we use metaphors and the like in our classrooms. A problem here, however, is that the metaphors or models we use are not coherent. They are far from pointing in the same direction, and some are even inconsistent or meaningless.

Take multiplication as one example out of many. Often multiplication is introduced via a model of repeated addition: $3 \cdot 5$ is 3 (apples) added to (another) three (apples) etc. 5 times. Of course this model must be abandoned, and may even cause conceptual problems when the child somewhat later in school is introduced to negative or rational numbers. Adding a number to itself a negative number of times, or $\frac{5}{13}$ or π times, makes little sense.[11] Thus other models or metaphors, using ice cubes or geometrical intuition, written and mental algorithms, methods of factorization, etc., are introduced along the way. In all cases these methods grossly underdetermine the concept of multiplication,[12] and may even be mutually contradictory. But children do learn multiplication, anyway most do.

Following Jenkins (2008) this is so, since mathematical concepts are empirically grounded. Empirical grounding is also a way to explain why mathematics is learnt via abstraction and generalisation, beginning in everyday reality. There must, however, be more to it than that. Let us again take an example, this time from geometry: The angular sum in a triangle is, we are taught, 180°. In classrooms this fact is, at least in Swedish classrooms, first motivated by letting the students cut off the corners of triangles they draw with paper and pencil, lay the little pieces of paper side-by-side, and note that they all more or less form a straight angle.[13] In the best of worlds, this empirical 'demonstration' is later complemented with a more or less formal proof. Now, why should we need a formal proof of a fact that we have already demonstrated empirically? And how is it at all possible to demonstrate an empirical fact deductively?

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[10] Abstraction and generalization have been treated by many: Aristotle, Berkeley, Locke, Piaget, Dewey, and Sfard just to mention a few. For references and discussions see e.g. von Glasersfeld (1991).

[11] In fact, seeing multiplication as repeated addition causes problems already for the restricted case of natural number. This is also discussed by Steiner (2005).

[12] It may be noted here that multiplication is not at all definable in terms of addition within a framework of first order logic – both the arithmetic of only addition (Presburger Arithmetic) and the arithmetic of only multiplication (Skolem Arithmetic) are decidable, while the combined arithmetic of addition and multiplication (First Order Arithmetic) is not.

[13] In Sweden this is even the *only* technique used in some textbooks for year 10!

In the following we present an idea of concept formation in mathematics that highlights the role of mathematics in empirical applications, and which tries to answer these questions. Hopefully, we make it clear how the applicability of mathematics can be explained, and how this view may affect mathematics education. Note that we do not present a theory of how individuals acquire concepts, but an idea of how mathematical concepts find their way into mathematics. Thus our view is philosophical. An attempt to use these ideas psychologically, to analyse how individuals reinvent concepts, is provided by Dawkins (2015). A framework for discussing the relationship between mathematics and reality in a 'situated learning'-context is provided by Dapueto and Parenti (1999). Here we start out with a few words on the concept formation process in mathematics.

Concept Formation in Mathematics

Two main philosophical sources for our views on concept formation are Aristotle's philosophy of mathematics, and Carnap's use of explications as a means to develop exact and fruitful concepts. Since these ideas have been presented elsewhere by Sjögren (2011) and Bennet and Sjögren (2013), and our focus here is on education, we will be rather brief.

According to Aristotle mathematical entities are inherent in substances, i.e. in individual objects. By abstraction these entities, or traits, can be isolated in thought, but they do not have separate existence like the Platonic Forms have according to Plato. Mathematical objects, according to Aristotle, are not pure forms, and they are not sensible objects, but they are separable from sensible objects in thought. In a process of abstraction the mathematician eliminates non-essential attributes in favour of essential ones. There is a difference, here, according to Aristotle, between physics and mathematics, in that the former treats accidental properties, like snub nosed, while the latter treats essential ones, like curved, (concerning Socrates's nose) (Lear, 1982; Ross, 1924). In Aristotle definitions are made via genus and differentia specifica to finally reach the essential attributes, eliminating the non-essential ones, or to reach those attributes we want to pay attention to in a certain context (Heath, 1998).

Now, an explication can be seen as an ontologically neutral device to accomplish something similar. A mathematician engaged in applied mathematics may try to isolate traits in objects or problems in order to receive essential ones, and by 'essential' we do not mean essential in a metaphysical meaning, but traits that may make it possible to analyse the object or problem at hand mathematically.

Let us, before turning to Carnap's views , briefly sketch just two examples of historical sequences of explications of concepts, central in teaching mathematics – function, and area. We begin by treating function.

The first explicit, algebraic definition seems to be due to Johann Bernoulli in 1718, and Euler, somewhat later, defined this concept in the following way:

> A function of a variable quantity is an analytic expression composed in any manner from that variable quantity and numbers or constant quantities. Quoted from (Kleiner, 1989, p. 284)

What Euler meant by "analytical expression" is not exactly clear, although he certainly meant it to be one formula. The concept function does not, however, have its origin with Bernoulli and Euler. It was used in the seventeenth century, by e.g. Leibniz, in geometrically formulated problems speaking of a tangent as a function of a curve. There are also historians who trace the concept all the way back to Ptolemy's tables of chords in the *Almagest*, or to his predecessor Hipparchus. The function concept can be thought of as the vague concept of one entity determining another in some kind of (possibly causal) dependency, and different early explications can be seen as ways of mathematizing this concept. With Fourier and Dirichlet we have the conception of an arbitrary, not law-like, connection between the variables, and from this idea the set-theoretic concept of function as graph emanates. The function concept has thus evolved from a geometrical one, via an algebraic concept, to the set-theoretical concept.[14]

Again there are several ways to teach students basic facts about functions.[15] In our experience a normal strategy is to start from e.g. length, with students in the class as arguments for a variable and their respective lengths as function values. Also used are the 'function machine' giving predictable outputs for given inputs (e.g. a squaring machine), input-output tables, graphs, functions as rules, and other representations. Of course these representations are used as models or metaphors. No one is to believe that functions really are any of these things, but the now standard, university text book notion of function as a type of relation, i.e. as a certain type of set, is far from the models used in (secondary) class. In spite of this, at least some students really do get appropriate conceptions of function. How is this possible?

Our next example is the *area* concept. In *Elements*, there are no arithmetical results on how to compute an area. Instead, Euclid, for a given geometrical object like a rectilinear figure showed how to construct a square with the same area (Euclid, 300 BCE/1996: Book II, prop. 14). Somewhat later Archimedes used techniques, anticipating integration, to study areas of different geometrical objects as, e.g., the area of the region enclosed by a circle. In the early seventeenth century Fermat, among others, could determine the area under a curve like $y = x^2$. This led in the hands of Newton and Leibniz to the infinitesimal calculus making it possible to define areas in a wider perspective. To go on, different integration techniques (as Lebesgue integration) together with measure theory were developed to be able to treat still more complex problems.[16]

In primary school the introducing of area is preceded by presenting and discussing the very idea of measurement. Maybe one first introduces direct measuring (the number of feet of a classroom wall), followed by indirect measuring (using a ruler). One next introduces the idea of measuring a rectangle, first using direct comparisons, and next using indirect measuring via plastic (or so) identical squares. Piaget's (1954) notion of conservation of area is, of course, of importance here. Further one connects

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[14] See Kleiner (1989) on the evolution of the concept function, and Kline (1972) for a more comprehensive treatment.

[15] See Vinner and Dreyfus (1989) for an early discussion on students' conceptions.

[16] See Kline (1972) for a comprehensive treatment.

area to lengths, and ends up with formulas for the areas of rectangles, circles, *etc.*[17] Now, how is it possible for students, presented to various metaphors and measuring techniques in different contexts, to grasp the mathematical (abstract) concept area?

Explications in Carnap's Sense

Our suggestion is that the processes of making abstractions, exemplified above, of isolating essential traits, using the nomenclature of Aristotle, is to be regarded as a process of making explications in the sense of Carnap (1950). In an explication the *explicandum* is the more or less vague concept, and the new, more exact one, is the *explicatum*. Here we use the term "exact" in the way Carnap does in the quotation below, synonymously with "precise" and as opposed to "fuzzy" or "vague". As an example Carnap mentions Frege's and Russell's explication of the cardinal number three as the class of all triplets. The criteria an explicatum must fulfil are as follows:

> 1. The explicatum is to be similar to the explicandum in such a way that, in most cases in which the explicandum has so far been used, the explicatum can be used; however close similarity is not required, and considerable differences are permitted.
>
> 2. The characterization of the explicatum, that is the rules of its use [...], is to be given in an exact form, so as to introduce the explication into a well-connected system of scientific concepts.
>
> 3. The explicatum is to be a fruitful concept, that is, useful for the formulation of many universal statements (empirical laws in the case of a nonlogical concept, logical theorems in the case of a logical concept).
>
> 4. The explicatum should be as simple as possible [...]. (Carnap, 1950, Ch. 1)

Formulating explications may be understood as a non-metaphysical way of constructing more precise concepts for a given purpose. In this way it is possible to replace vague, imprecise, or otherwise non-clear concepts by more exact ones. To be able to mathematize a part of reality, an abstract mathematical one or an empirical one, sufficiently exact mathematical concepts are needed. And this can be achieved as above by explication. Not only concepts of empirical science have an origin in reality, but also mathematical ones do. Mathematics has in this way been able to generate concepts via explications, concepts that are fruitful in mathematizing reality.

Examples are easy to find: The concepts mentioned above, function and area, can be seen as far removed from empirical reality, but tracing their origins, in a kind of concept archaeology, will lead us back to a more or less non-clear empirical counterpart. The concept of functionality has its origin in one process uniquely determining another. Perhaps it may even be seen as a mathematical counterpart to causality. With the modern set-theoretical explication this origin is lost. Striving towards generality mathematics has, through history, also strived towards even more abstract concepts. This tendency has furthered mathematics

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[17] In general we believe that mathematics education in many cases more or less follows a historical development. But we do not, thereby, believe that each student must, or should, undergo the same process. Gravemeijer and Doorman (1999) discuss this interesting issue, as do Witzke et al (2016).

from empirical experience even more. We therefore need to make some comments on the abstract character of mathematical concepts.

Mathematical Concepts are Abstract

The dichotomy of the realm of objects in abstract and concrete, is fundamental for philosophy in general and for ontology and epistemology in particular. We will not, and need not, try to define this dichotomy precisely, but in agreement with a vast majority of philosophers we believe objects like thoughts and numbers to be abstract, while objects like trees and trams are concrete. As stated above, we are not concerned with ontological issues here. Thus it is irrelevant for our discussion whether e.g. the number five is considered to be mental or not. Of course this doesn't mean that there are no constraints at all involved when speaking of (mathematical) concepts. As an example, learning would be impossible to explain (or understand) without presuming concepts to have (at least) a certain amount of inter-subjectivity. It does make sense to speak of 'grasping *the* number π or *the* concept of function'. Teachers even assess students' grasping of such concepts.

We do, however, like to emphasize the abstract character of mathematical concepts. This will also show the need of idealizing situations (see below) in order to make applications of mathematics possible.

Consider, in order to stress some philosophical problems for mathematics, the following drawing – a play with a painting of Magritte:

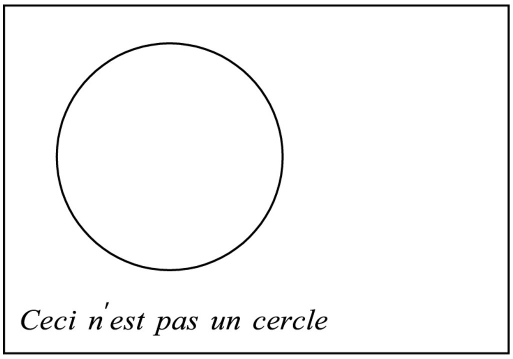

Figure 1. This is not a circle.

A student once told one of us that *π* gets different values using the inner and outer part of the 'circle', respectively. Ignoring that the diameter is different in the two cases, and believing that the drawing *is* a circle, this proposal is completely legitimate! Of course a drawing is not really a mathematical circle, since it has a width. Imagine a student who has grasped the idea of circle, and who gets the task of computing the diameter of a tree trunk. Naturally, they can raise the objection that a cut of the trunk isn't a circle. In fact, the step to imagine that a cut of the trunk is to be regarded as a circle isn't a trivial matter.[18] Here we regard the (precise) mathematical concept

[18] See Dapueto and Parenti (1999) for a discussion in a somewhat different context.

as an explication, in Carnap's sense, of the (vague) every day one. This example is not unique, and one could compare with the concepts velocity and work in physics and in ordinary language situations. Examples from other fields are fish and species in biology (are octopuses fish?), salt and equilibrium in chemistry, and mental illness in psychiatry and law.

In the commentaries of the Swedish National Agency for Education concerning the Swedish curriculum, it is emphasised that students are to "meet natural numbers" (Swedish National Agency for Education, 2011). What does it mean to 'meet' the number three? Suppose that we with Lewis (1986) or Maddy (1990) can experience small sets of concrete things (and not just the things themselves), and that our attention is directed towards a certain collection of objects – be it, e.g., three coffee cups with saucers and spoons. How are we to know which number(s) we 'meet' in a situation like this?

Concerning geometry, the same commentaries tell us that

> The area of knowledge "Geometry" deals with how to measure and describe ones vicinity. Within geometry one recognizes, measures, interprets, and describes the world around us from a spatial perspective using different forms of expression. (Swedish National Agency for Education, 2011, p. 18, our translation)

This purely empirical perspective is further emphasized when we learn that students in the lower grades shall "meet geometric objects such as boxes and balls", and that they shall "draw and build" rectangles and "construct regular patterns in the world around them".

The same commentary has suddenly switched perspective, however, when writing about lower secondary school (year 7 to 9). Here the student should be given the opportunity to "argue for the correctness of formulas", reason about how one "within mathematics decide on what is true", and that a foundation is to be given for the students to understand the concepts proposition and proof.

Now, obviously, there are no propositions in geometry concerning boxes and balls, so somewhere between the lines in these passages, there is a tacit transformation from geometry as part of our empirical study and description of the world to geometry as deductive science. But how can we expect students to understand the purpose of proofs, when mathematical objects *are* objects in everyday life? Again: Why shall I prove The Pythagorean Theorem, when I have measured so many triangles?

We conclude this section with an example concerning probabilities. Consider the outcomes throwing a single die. When throwing a physical die several outcomes are possible. Depending on the form of the die, it can land on any of its faces, and if the corners and edges are well rounded it may even land on a corner. When describing this situation in mathematics, we speak of a mathematical die. This is a die that is perfectly homogeneous, has sharp edges and corners, for which the only possible outcomes are those with one of the faces up, and for which the probability that a specific face comes up is exactly 1/6. This is not a die that exists in empirical reality.

We can generalize the situation one step further, and describe it as a discrete, uniform, probability distribution with six outcomes. And now there is no difference, mathematically speaking, between drawing a card from a deck of six cards or drawing a ball from an urn with six balls in it.

Whatever ontological status mathematical concepts may have they are, thus, abstract. This means that we do not *meet* mathematical entities, neither circles nor numbers, in our daily experiences, but we can, as Aristotle suggested, via a process of abstraction isolate the essential traits of a situation (Franklin, 2014; Sjögren, 2011). This also means that introducing new concepts in mathematics education may be problematic.[19] Mathematical concepts have, however, an origin in empirical experience. As stated above they are abstractions, carved out via explications. This makes it understandable how the applicability of mathematics is possible. Still, this is complicated, and it is somewhat worrying that it isn't discussed in e.g. the Swedish mathematics curriculum. In these texts the applicability of mathematics is taken for granted, and not a word is said on the problem of relating mathematics to 'reality' – a well-known (and important) problem in the philosophy of mathematics as well as for mathematics education.

If, on the other hand, mathematics curricula would explicitly reflect on mathematical concepts being abstract *and* having an empirical origin, mathematics being applicable *since* it is abstract, this could give teachers guidance as to why and how they may use concretizations without pretending mathematics to actually be *about* everyday objects. As e.g. Witzke et al (2016) show, pretending mathematics to be about concrete objects makes for an "abstraction shock" when students meet a more scientific view of the subject.

Abstractions and Idealizations

Related to the process of abstraction is the process of idealization. Speaking in the language of Aristotle, the process of abstraction can be seen as an elimination of non-essential properties; properties that we do not want to pay attention to, to follow Lear (1982). Triangles can be, e.g., isosceles or right-angled, and in a process of abstraction we may disregard features such as these. When studying composition of functions we leave out of account non-essential traits such as if the functions are odd or even and arrive at, e.g., their group structure. Non-essential properties of an object of learning may, and should, be varied in learning situations, while, of course, keeping essential properties fixed (Ling & Marton, 2012; Marton, 2014).

In a process of idealization, on the other hand, the mathematician or empirical scientist may disregard essential properties such as friction when studying mechanical systems. In cases such as these the aim is rather to arrive at a problem description that can be analysed using mathematics at some suitable level.

Another road to understanding the difference between abstraction and idealization is implicit in Lewis' ideas of possible worlds connected to modal logic. To cite Lewis:

> Idealizations are unactualized things to which it is useful to compare actual things. An idealized theory is a theory known to be false at our world, but true at worlds thought to be close to ours. The frictionless planes, the ideal gases, the ideally rational belief systems – one and all, these things that exist as parts of other worlds than our own. The scientific

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[19] See Bennet and Sjögren (2013) for a discussion.

> utility of talking of idealizations is among the theoretical benefits to be found in the paradise of possibilia. (Lewis, 1986, p. 27)

These ideal objects thus exist in possible worlds close to ours, but not in our world. But abstract objects, at least mathematicalia, are to be applicable in every possible world. Accepting this jargon of modalities might shed some light on the distinction. When a mathematician or a scientist makes an idealization, they are aware of the introduction of falsehoods. In making abstractions, on the other hand, focus is on arriving at essential traits of the studied objects, and the realistically inclined mathematician might think they have analysed some existing object, as when numbers are explicated as properties of classes. The anti-realist mathematician can make the same abstraction, but even though regarding the received concept as fruitful, they refuse to speak of the abstract objects as having an independent existence.

Still another way of looking at the difference between idealization and abstraction, when already developed mathematical theories are used in applications, is that we try to idealize the physical situation so that it fits into the mathematics. In a first approximation we think of, e.g., frictionless planes or non-elastic collisions. These approximations may be taken as limit situations of existing states. We can imagine surfaces with less and less friction, and ideally we can imagine frictionless surfaces. The case with abstraction, at least when trying to find new concepts, is different. Here we try to see what is essential in the given problem situation, or to see patterns in structures that can be developed into fruitful mathematics as when Newton defined, or discovered, that velocity may be explicated as a time derivative, or that force is proportional to the second derivative of position. Roughly speaking, with an idealization we try to fit reality into mathematics, mathematizing horizontally, and with an abstraction we try to isolate essential features in order to explicate concepts.

Mathematics and the world

We now return to the problem of applicability of mathematics in mathematics education. Taking the idea of concept formation described above ad notam, and connecting it with the need to idealize situations, will shed light on how mathematics is applied in school.

To apply mathematics in a 'real world' situation, a generalization is needed. This is so even in very simple situations. A student using the same formula when calculating the volume of a classroom and the volume of a swimming pool, needs to realize, not that classrooms or swimming pools are actually cuboids, which they of course are not, but that they, idealized, have a common shape, and that this shape is a mathematical entity which volume may be calculated with a certain formula.[20]

A similar comment is in place concerning mathematical modelling. Elaborating the metaphor of Niss (2012) for modelling as a triple (D, f, M), where D is a real world domain, M a mathematical structure, and f a (structure preserving) mapping from D to M, idealization involves the choice of D. In fact a more accurate metaphor would have five components (R, g, D, f, M), where R is the actual real world situation, D is an ideal-

[20] Steiner (2005) gives other examples. See also Blum and Niss (1991).

ization of *R*, *g* is a (forgetful) mapping from *R* to *D*, *f* is a structure preserving mapping, and *M* is a mathematical structure. The idealization thus involves the choice of *D* related to (the given – also this is of course a simplification) *R*, while abstraction concerns the choice of *M* (and *f*).[21]

Thus an understanding of the interplay between abstraction and idealization is an important key, if we are to explain the difficulties with the applicability of mathematics in advanced science as well as in school.

Turning to the classroom situation, a way to treat applications in mathematics teaching at all levels is to have extensive discussions in the classroom on issues of idealizations. One and the same situation can typically be idealized in different ways. In making these idealizations, considerations must, of course, be taken to the mathematical level of the students. Note that we *do not* treat the *real* situation, but an idealization of it.[22]

Using Carnap's framework for theoretical concepts as explications one may view e.g. (instantaneous) velocity as an explication of our everyday concept speed, work in physics as an explication of our everyday work, circle in mathematics as an explication of our everyday circle or ring, and so forth. Note, however, that most explications are theory dependent, and vague everyday concepts may be explicated in different ways. Thus, velocity is a different concept in classic Newtonian physics, compared to relativity theory. In the first case velocity is an additive concept, in the second case not. But in both cases velocity is closely connected to our empirical world.

Mathematical concepts, however, are different from empirical ones. Concepts such as elephant or green may be learnt by pointing out elephants and non-elephants or green and non-green objects in our surroundings, but a concept like multiplication, or number (finite cardinal) for that matter, is not possible to point out in the same manner. Thus we use metaphors and the like, and help learners to explicate and abstract. In case the concept to be learnt is mathematical, it may be argued that essentially only one explication is possible, and learners may succeed in grasping the concept in spite of the teacher using underdetermining or even conflicting metaphors.[23]

However this may be, there is a distance from our experienced world to the abstract 'world' of mathematical concepts (not necessarily meant in a Platonic sense). Mathematical concepts are not empirically grounded, as concepts in empirical sciences are, even though they may have an empirical origin (Jenkins, 2008; Lakoff & Núñez, 2000; Sjögren & Bennet, 2014). Instead the very idea with mathematics is that it is abstract, general, and non-contextualised. Geometry is a good example here: As far as geometry describes the world, it is not mathematics, i.e. in so far as it is mathematics, it doesn't describe the world. Geometry as mathematics is deductive and presents to us a number of possible worlds via different geometries, each geometry having its own class of models (logically speaking). It is then up to the physicist to decide which geometry that best suites their purposes.[24] The

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21 Also cf. Blum and Niss (1991).

22 This issue is also discussed by Bråting and Pejlare (2008).

23 This 'robustness' of mathematical concepts is discussed in Bennet and Sjögren (2013), Sjögren (2011), and Sjögren and Bennet (2014). Dawkins (2015), drawing on Sjögren (2010), discusses a related notion of "psycological explication".

24 For a discussion of these issues, see Giaquinto (2007).

same goes, in fact, for arithmetic. In principle it is possible to choose some non-standard model of natural numbers as the adequate description of the world. Here, however, there is a difference from geometry, since there is a categorical description of the standard model in the form of the Dedekind-Peano second order number theory. In light of this theory, we are not free to choose truth-values for propositions that concern finite cardinal numbers.

A way of putting this is that mathematical truth makers are internal, while truth makers for empirical sciences are external. Thus the notion of proof is not only essential in mathematics, but proofs are the only way to verify truths. Using a Popperian terminology, paper triangles and the like belong in a context of discovery, while proofs belong in a context of verification. Both contexts are present in the classroom, but they should not be mixed up.

Thus it is important in learning situations that the teacher (and learner), both idealizes and abstracts. To idealize is, in a sense, to substitute an appropriate possible world for the real world, and abstraction helps finding the appropriate mathematical concepts for the particular context at hand. Abstractions may be seen as explications in Carnap's sense, and we believe it is possible for students from an early age to abstract. Let us give here a simple example that we have tested.

Students may, already in the first years of primary school, be given the following task: Let's say that you may climb a staircase taking small steps, i.e. one step at a time, or big steps, i.e. taking two steps at a time. In how many different ways can you reach the twentieth step by combining small and big steps?

We have noticed that this is a type of task that really engages children – they see it as a game with very simple rules. What happens is that, given this task to small groups, students aged 7 or older (including groups of university students) start walking in the nearest staircase. They soon realize that it is hard to keep track of what combinations are tested, so they start recording them using paper and pencil. The next step is to stop walking, and stick to record keeping. Finally they loose interest in the staircase altogether, and work only with numbers. Even the youngest students fairly quickly see the pattern, and the final step is to just prolong the number sequence (which is the Fibonacci sequence). Finally, at least with students from age 9 or 10, it is possible to discuss how this series is defined, and precisely why it solves this particular problem (the explanation being, indeed, simple).

This example shows how it is possible to go from a concrete, everyday problem (a game of walking in stairs) to pure mathematics (discussing number series), even in the beginning of primary school. And this without pretending that mathematics is "met" in or is part of everyday life.

Using a terminology from Gravemeijer (1999) and Zandieh and Rasmussen (2010) the students' activities may be described as a process from an initial situational stage of real world experiment through a referential stage of structuring their game, onto a general stage where mathematics takes form, and reality is put in the background, ending up in a formal stage where the students discuss, in a sense, pure mathematics.[25] Our thesis is that the mathematical concepts link to the initial real world situation by being explications of real world concepts.

[25] For a discussion also cf. Dawkins (2015).

Conclusions

In contrast to Lundin (2012) we thus see no reason to pretend that word problems in school are to give an illusion of being 'real world' problems in any other respect than being examples of how mathematics may be used as a tool in combination with a process of explication, abstraction, and idealization. The word problem functions as a starting point in a transition from discussing a real world problem, to problem solving *within* mathematics.

These processes constitute the very core of mathematics, making mathematics, in the famous words of Wigner (1960), unreasonably effective in the natural sciences. Thus mathematics is not applicable *in contrast to* being abstract, but applicable *since* it is abstract. Thereby, in school, in order for students to learn mathematics, teachers must lift their mathematics teaching from the concrete everyday world to the abstract and general. They must lead the students from real world concepts to explications in the form of appropriate mathematical concepts. It is not until this is done, that mathematics becomes understandable as a deductive, yet applicable, science.

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